DETERMINATION OF VARIABLE COEFFICIENT OF HEAT TRANSFER FOR A THIN SEMIINFINITE ROD

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The heat transfer in a semiinfinite rod, cooled from the lateral surface according to a time-dependent law, is investigated. The law of heat transfer is found from the given temperature and the temperature gradient at the end-face of the rod.

In the problem

$$\left[\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}+\gamma(t)\right]T=0,\ 0\leqslant x<\infty,\ 0<t<\infty; \tag{1}$$

$$T|_{x=0}=\vartheta(t); \tag{2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0}=q(t); \tag{3}$$

$$T|_{x=\infty}=0; \tag{4}$$

$$T|_{t=0}=0, \tag{5}$$

from given values of the functions $\vartheta(t)$ and $q(t)$ it is required to determine the variable heat transfer coefficients $\gamma(t)$. This problem describes, for example, the cooling of a thin semiinfinite rod by a liquid flux with changing velocity or temperature.

After the well known substitution $\theta = T\exp(\int_0^t \gamma dt)$, instead of (1) we have

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\theta=0. \tag{6}$$

From here we can obtain the relationship among ϑ, q, and γ (for example, using the method discussed by us earlier [1, 2]) in the form

$$q\exp\left(\int_0^t \gamma dt\right)+\frac{d^{1/2}}{dt^{1/2}}\left[\vartheta\exp\left(\int_0^t\gamma dt\right)\right]=0, \tag{7}$$

where we have used the fractional differentiation operator

$$\frac{d^{1/2}f(t)}{dt^{1/2}}=\frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t\frac{f(\tau)\,d\tau}{\sqrt{t-\tau}}, \tag{8}$$

$$\frac{d^{1/2}t^{\mu}}{dt^{1/2}}=\frac{\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{1}{2}\right)}t^{\mu-\frac{1}{2}},\quad \mu>-\frac{1}{2}. \tag{9}$$

Let the functions ϑ and q be given in the form of power series of $t^{1/2}$

$$\vartheta=\sum_{n=0}^{\infty}a_n t^{n/2};\quad q=\sum_{n=0}^{\infty}b_n t^{(n-1)/2}. \tag{10}$$

Then the solution can be constructed in the form of the series

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$$\exp\left(\int_0^t \gamma dt\right) = \sum_{n=0}^{\infty} c_n t^{n/2}. \qquad (11)$$

Substituting (10) and (11) into (7) and equating the coefficients of equal powers of $t^{1/2}$ we successively determine all c_n. This is possible under the condition

$$a_0 + \sqrt{\pi} b_0 = 0. \qquad (12)$$

Condition (12) is easy to explain physically. At the initial instant of time the law of cooling $\gamma(t)$ must not depend on the relationships between ϑ and q. Here c_0 is an arbitrary quantity and the remaining constants are expressed in terms of c_0 in the following way:

$$c_1 = -\frac{a_1\Gamma(3/2) + b_1}{a_0\Gamma(3/2) + b_0} c_0 = \alpha_1 c_0,$$

$$c_2 = -\frac{[a_1\Gamma^{-1}(3/2) + b_1] c_1 + [a_2\Gamma^{-1}(3/2) + b_2] c_0}{a_0\Gamma^{-1}(3/2) + b_0} = \alpha_2 c_0, \qquad (13)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_n = -\frac{\sum_{k=1}^{n} c_{n-k}\left[\Gamma\left(\frac{n+2}{2}\right)\Gamma^{-1}\left(\frac{n+1}{2}\right) a_k + b_k\right]}{a_0\Gamma\left(\frac{n+2}{2}\right)\Gamma^{-1}\left(\frac{n+1}{2}\right) + b_0} = \alpha_n c_0.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

It is evident that arbitrary constant c_0 occurs in all c_n only in the form of a factor, so that $c_n = \alpha_n c_0$. Therefore, in taking the logarithm and in differentiation of (11) c_0 drops out and the final solution has the form

$$\gamma(t) = \left(\sum_{n=1}^{\infty} \frac{n}{2} \alpha_n t^{(n-2)/2}\right)\left(1 + \sum_{n=1}^{\infty} \alpha_n t^{n/2}\right)^{-1}. \qquad (14)$$

Example. Let $\vartheta = a_0$, $q = -a_0(\pi t)^{-1/2} + b_1$. From (13) we get

$$\alpha_n = -(b_1/a_0)\left[\Gamma\left(\frac{n+2}{2}\right)\Gamma^{-1}\left(\frac{n+1}{2}\right) - \frac{1}{\sqrt{\pi}}\right]^{-1} \alpha_{n-1}. \qquad (15)$$

Since $\lim_{n\to\infty} |\alpha_{n-1}/\alpha_n| \sim O(\sqrt{n})$, the series in (14) (for (15)) converges absolutely for all values of t.

NOTATION

α, a, b, c	are the coefficients in the power series;
T	is the temperature;
ϑ	is the temperature at the end-face of the rod;
q	is the temperature gradient at the end-face of the rod;
x	is the coordinate;
t	is the time;
γ	is the heat-transfer coefficient;
θ	is the auxiliary variable;
f	is the arbitrary function;
μ	is the index of the power function;
n, k	are the summation indices.

LITERATURE CITED

1. Yu. I. Babenko, Certain Problems Frequently Encountered in the Theory of Nonstationary Combustion, in: Combustion and Explosion [in Russian], Nauka (1972).
2. Yu. I. Babenko, Use of Fractional Derivatives in Problems of Heat Transfer, Proceedings IV All-Union Conference on TMO, ITMO AN BSSR, Vol. 8, Minsk (1972), p. 541.